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Newton's Method

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Abstract

Newton's Method

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Root-finding algorithms have been studied for ages for their various applications. Newton's Method is just one of these root-finding algorithms. This report discusses Newton's Method and aims to describe the procedures behind the method and to determine its capabilities in finding the zeros for various functions. The possible outcomes when using this method are also explained; whether the Newton function will converge to a root, diverge from the root, or enter a cycle. Modifications of the method and its applications are also described, showing the flexibility of the method for different situations.

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Chapter 1: Introduction

In our current educational system, high school algebra classrooms focus on functions by using technology to generate models, tables, and graphs [**3**, p.125]. Highly advanced calculators assist with students' learning by allowing students to more easily visualize what they are working with. While students are exploring classical functions such as linear and quadratic (for Algebra I), inverse, absolute value, square root, and logarithmic (for Algebra II), they are expected to describe certain characteristics of the functions such as rates of change, intercepts, zeros, asymptotes, and local and global behavior such as maximums and minimums [**4**].

Just knowing the characteristics of functions is not enough. Students must be able to interpret what characteristics a question is asking about within a word problem. Being able to locate and describe the x-intercept is a large portion of quadratic questions within Algebra I and II. Finding solutions, or x-intercepts, of functions is not something new. Mathematicians have been doing it for a long time [**1**, p. 584]. Although there are a multitude of ways of finding these solutions, there are some specific methods that are used with certain functions. For example, the quadratic formula is a method that can only be used to help find the roots of quadratic functions. If the functions are simplistic in nature, such as linear functions or quadratics where the roots are integers, the solutions can be located by graphing to determine where the graph crosses the x-axis or by viewing the table of values. Other functions which are more difficult and cannot be easily viewed would require approximation methods to get as accurate as required to the solutions. One of these approximation methods is called the *bisection method*.

Another method is called *Newton's Method* or *Newton-Raphson Method.* The equation for Newton's Method,

$$N = x - \frac{f(x)}{f'(x)} \tag{1}$$

requires some knowledge of calculus so that one may find the derivative of a function, $f'(x)$. Newton's Method is an iterative process which requires an initial value in finding the roots of a function. The method follows an algorithm where values are repeatedly produced and evaluated as many times as necessary until an acceptable approximate value for the root is found. In order to use this method, one must understand what an *iteration* is and how to iterate.

Chapter 2: Iterations

What is an *iteration*? When one iterates, the process is repeated multiple times. There are countless examples of iterations that humans do on a daily basis. One basic example would be walking since one repeats the process of placing one foot in front of the other multiple times in order to walk.

The results from iterating functions are called the iterations. With mathematics, specific functions are iterated to find orbits, roots, fractals, etc. In order to find roots, one can use Newton's Method and each iteration approaches the root with a better approximation than the previous iteration did.

The beginning value for the iteration is called the *seed value*, denoted generally as x_0. The next input value would be the output of the seed called x_1. This process is iterated *n* times where $n = 0,1,2,3,4, \ldots$ and terminates when a specific goal is met, which depends the original purpose of the iteration. When finding roots, the process would terminate when the zero is found or when one has arrived at an appropriate approximation [**2**, p.393].

An example of an iteration being performed on a function is demonstrated in Table 1. The function$f(x) = 3x$ has been iterated five times with the starting seed value $x_0 = 2$. The first iteration produces $x_1 = 6$. Using this as the new input, the new output is now 18. The process is repeated and can be represented by $f(x_n) = x_{n+1}$ where $n = 0,1,2, \ldots$.

Table 1: Example of iteration

	f(x)=3x	
Iteration	Input	Output
x_0	2	6
x_1	6	18
x_2	18	54
x_3	54	162
x_4	162	486
x_5	486	1458

The set of iterates $\{x_0,\ x_1,\ x_{2,\ldots}\}$ is called the *orbit* of point x_0 [**9**, p. 252]. The orbit for $f(x) = 3x$ as seen in Table 1 can be described as *increasing* since all the outputs are constantly increasing with each iteration. Using different seed values will produce different orbits. If $x_0 = -5$, the orbit would be *decreasing* since each iteration would be tending towards negative infinity. There can also be orbits that are *periodic*; that is $f^k(x) = f^{k+p}(x)$ for some positive integers k and p [**9**, p. 252].

When performing iterations from a graph, one can draw the line $y = x$ to assist in determining the iterations. Since, with each iteration, the new input is the previous output, the line $y = x$ can be referenced to see what the next iteration will result in. The process starts at the seed value, then making a vertical line to $f(x)$, then a horizontal line to $y = x$ [**10**, p 180]. Illustration 1 demonstrates this process.

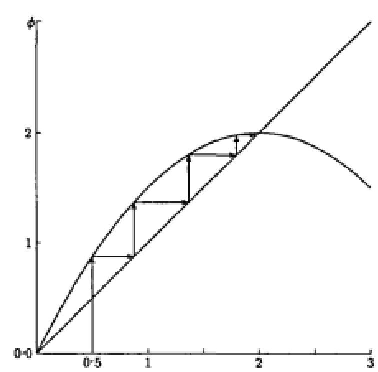

Illustration 1: Graphed iteration [**10**, p. 180]

Chapter 3: Newton's Method

Newton's Method originated in the year 1669 [**5**, p.1087]. Newton was only finding roots of polynomials and was not using the method which is seen today. Newton's original method did not contain any derivatives requiring calculus. Instead, a series of polynomials were created to find the best approximation for a root. In order to find the roots of a function f, one evaluates the function at $x + h$, where x is some initial guessed value, and sets it equal to 0. Once the function is evaluated, the higher order h terms are disregarded. Next, the equation is solved for h. This h value is then added to the initial x-value in order to provide a better approximation for the root of the function f.

An example using Newton's original method without calculus is applied to the function $f(x) = x^3 - 2x - 5$. To solve the equation $f(x) = 0$, one starts with an initial x-value of 2 and evaluates the function at $f(2 + h)$. Therefore,

$$f(2 + h) = (2 + h)^3 - 2(2 + h) - 5 \tag{2}$$

$$0 = 8 + 12h + 6h^2 + h^3 - 4 - 2h - 5 \tag{3}$$

$$0 = 10h - 1 + 6h^2 + h^3. \tag{4}$$

The higher h terms are disregarded which leads to

$$0 = 10h - 1 \tag{5}$$

$$h = 0.1. \tag{6}$$

This h value of .1 is added to the initial x-value to produce a better approximation of 2.1. By ignoring the higher order h terms, Newton was able to get a better approximation for the root. The process could be iterated further to get a better approximation.

In 1690, J. Raphson introduced the current form of the equation for Newton's Method with the derivative of $f(x)$ [**5**, p. 1087],

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{7}$$

One can verify that this method will arrive at the same approximation as Newton's original method. Using the same function as before and the same starting x value of 2,

$$N = 2 - \frac{f(2)}{f'(2)} \tag{8}$$

$$= 2 - \frac{8-4-5}{12-2} \tag{9}$$

$$= 2 - \frac{-1}{10} \tag{10}$$

$$= 2.1 \tag{11}$$

Both ways arrive at the better approximation of 2.1 for the root value. This method is sometimes called the *Newton-Raphson Method* since Raphson introduced it using the derivative function in the method.

Illustration 2 demonstrates (7) where $f(x)$ is some function. Where the function value associated with the seed x_0 is located on the function f, a tangent line can be generated. This tangent line intersects the x-axis closer to the root than x_0 did. This closer approximation is called x_1. Each tangent line is getting closer to crossing the x-axis where the original function does [**10**, p.185].

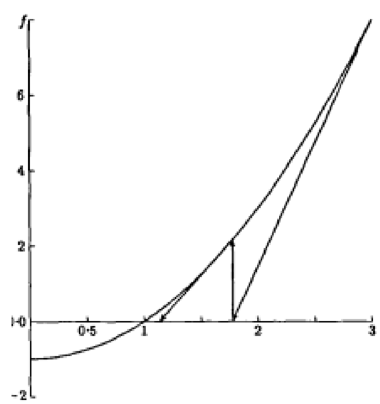

Illustration 2: Newton's Method [**10**, p 185.]

How can the value of x_1 be found? Since it is on the x-axis, the point is $(x_1, 0)$. The slope of this line is known to be $f'(x_0)$. Using the point-slope formula and solving for x_1,

$$0 = f(x_0) + f'(x_0)(x_1 - x_0) \tag{12}$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} \tag{13}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{14}$$

Newton's Method requires the function f to be a differentiable function $f: R \to R$ in order to locate the roots of a function by using the iterative method described by (7). The original root approximation, the seed value, is inserted into (7) and is iterated multiple times until a close enough approximation is acquired.

Table 2 shows Newton's Method for finding the roots of the function $f(x) = x^2 - 5x$. With the starting seed value $x_0 = 4$, four iterations are required in order to arrive at the root, $x = 5$.

Table 2: Example of convergence (to the 8th decimal place)

x	f(x)	f'(x)	Aprox. Root
4	-4	3	5.333333333
5.333333	1.777777778	5.666666667	5.019607843
5.019608	0.098423683	5.039215686	5.000076295
5.000076	0.000381481	5.00015259	5.00000000

Newton's Method is popular because of its higher *convergence rate* compared to other methods. Convergence rate for a sequence $\{x_n\}$ converging to a value x^* is defined as

$$|x_{n+1} - x^*| \le \lambda |x_n - x^*|^{\alpha} \tag{15}$$

where $0 < \lambda$ and α are real numbers, and α is the *convergence rate*. There are special names for when $\alpha = 1, 2, 3$: linear, quadratic, and cubic respectively [**9**, p. 252]. To prove quadratic convergence for Newton's Method when a function has a simple root, let x^* be the simple root of function f therefore $f(x^*) = 0$. Taking the Taylor series expansion about point x_n provides,

$$f(x^*) = 0 = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(x_n)(x^* - x_n)^2}{2} \tag{16}$$

through algebraic manipulations,

$$x_n - x^* - \frac{f(x_n)}{f'(x_n)} = \frac{f''(x_n)(x^* - x_n)^2}{2f'(x_n)}. \tag{17}$$

Substitution of (7) yields,

$$x_{n+1} - x^* = \frac{f''(x_n)}{2f'(x_n)}(x^* - x_n)^2 \tag{18}$$

where $\frac{f''(x_n)}{2f'(x_n)} = \lambda$.

Therefore, Newton's Method has quadratic convergence, the error is approximately proportional to the square of the previous error, given $f'(x_n) \neq 0$ and $f''(x_n)$ has some finite value [**10**, p. 185-186].

Chapter 4: Possible Outcomes

Newton's Method does not necessarily always converge to the roots of a function. There is the possibility of failing to locate the desired root. The possibilities when using Newton's Method are: convergence to a root, divergence, or entering an n-cycle period. These outcomes all depend on the function which is being operated upon and the initial seed value choice.

When using Newton's Method, one may encounter a function where the method produces a cycle. Table 3 shows how applying Newton's Method to the function $f(x) = 3x^5 - 10x^3 + 23x$ resulting in a periodic cycle where $n = 2$.

Table 3: Example with 2-cycle period

x	f(x)	f'(x)	Aprox. Root
1	16	8	-1
-1	-16	8	1
1	16	8	-1
-1	-16	8	1

Illustration 3 shows the 2-cycle period that is generated when applying Newton's Method. When $x_0 = 1$, a tangent line is generated that intercepts the x-axis at -1. This new input then generates the tangent line which intercepts the x-axis at 1. This 2-period cycle will continue indefinitely since the two tangent lines mirror each other. If $x_0 = 2$ had been chosen, the method would have successfully found the root at 0.

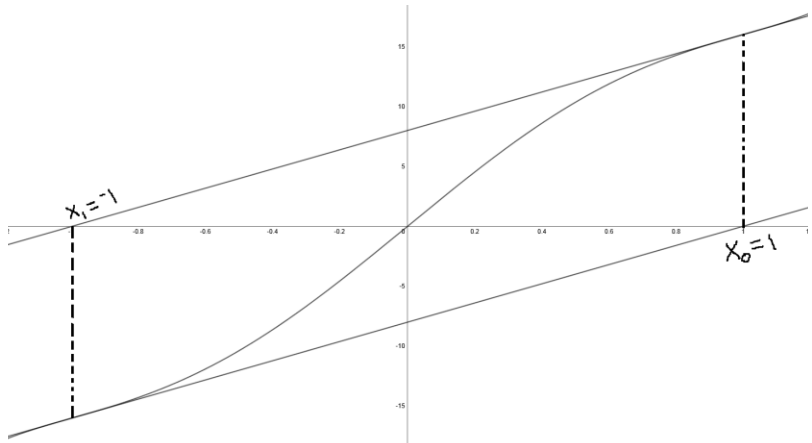

Illustration 3: Periodic cycle

Another situation which can occur is if the method diverges from the root towards infinity [**7**, p.1085]. Table 4 shows the results of applying Newton's Method on the function $g(x) = \frac{x}{1+x^2}$ with $x_0 = 0.75$. The root is at 0; but, with each iteration, the approximate root is increasingly more negative, which is further away from 0 than the previous one. This situation is illustrated in Illustration 4. If $x_0 = 2$, each iteration would diverge away from the root but towards positive infinity instead of negative infinity.

Table 4: Example of divergence (to the 8th decimal place)

x	g(x)	g'(x)	Aprox. Root
0.75	0.48	0.1792	-1.92857143
-1.92857	-0.40865	-0.122096	-5.27552935
-5.27553	-0.18298	-0.032279	-10.944297
-10.9443	-0.09062	-0.008143	-22.0728761
-22.0729	-0.04521	-0.00204	-44.2365474

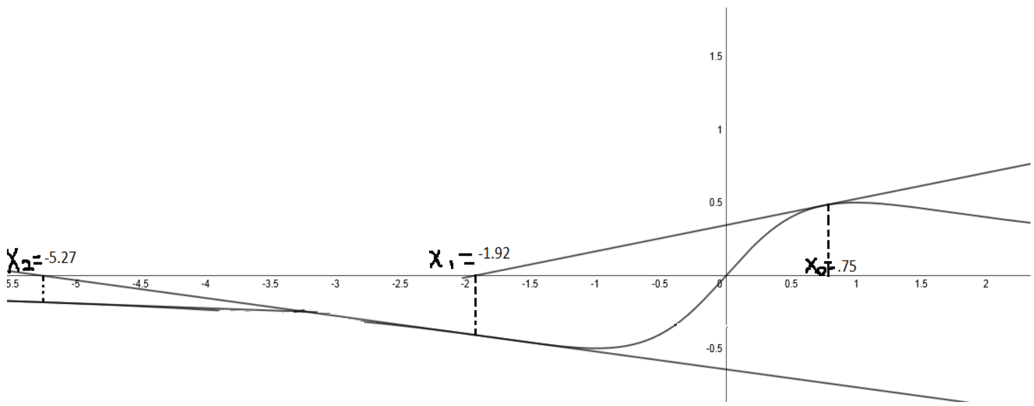

Illustration 4: Divergence

Another situation which can occur is if an evaluated point produces a horizontal tangent line to the function, which is generated due to $f'(x_n) = 0$ [**7**, p.1084]. Since a horizontal tangent line will never intersect the x-axis, the method will fail at locating the root of the function. If $x_0 = 1$ for function g(see pg. 10), a horizontal tangent line is produced as shown in Illustration 5.

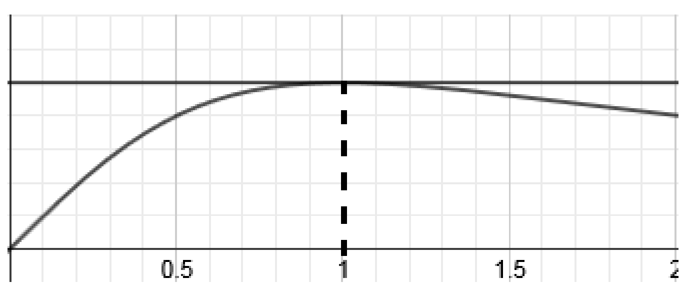

Illustration 5 Horizontal tangent line

Chapter 5: Modifications and Applications

Newton's Method can be modified and applied to solve many applications and different types of problems. Throughout the years, it has been modified and adopted by many mathematicians to solve a variety of problems [**7**, p.1084].

One of the most basic types of problems that Newton's Method can be used for is to solve a system of non-linear equations. Golbabai and Javidi demonstrate solving three different systems of non-linear equations with Newton's Method and show the results on how many iterations were required to locate the desired solution [**6**, p.549].

The system of non-linear equations used in this case has the form

$$\varphi(x) = \begin{cases} f(x), \\ g(x), \end{cases} x = (x, y) \in R^2, \tag{19}$$

where $f, g: R^2 \to R$ and $\varphi: R^2 \to R^2$. Assuming that $x_* = (\propto, \beta)$ is a zero of (19) and $q = (\lambda, \gamma)$ is an intial guess relatively close to x_* and taking a Taylor's series expansion around q yields

$$\varphi(x) = \begin{cases} f(q) + (x-\lambda)f_x(q) + (y-\gamma)f_y(q) + F(x) = 0 \\ g(q) + (x-\lambda)g_x(q) + (y-\gamma)g_y(q) + G(x) = 0 \end{cases}$$

where $F(x) = f(x) - f(q) - (x-\lambda)f_x(q) - (y-\gamma)f_y(q)$ and $G(x) = g(x) - g(q) - (x-\lambda)g_x(q) - (y-\gamma)g_y(q)$.

Two different algorithms are presented for solving such a system

$$z_{n+1} = z_n - J^{-1}(f,g)(z_n)\varphi(z_n), \quad n = 0,1,2,\ldots \tag{20}$$

and

$$z_{n+1} = z_n - J^{-1}(f,g)(z_n)\left(\varphi(z_n) + \begin{bmatrix} F(z_n - J^{-1}(f,g)(z_n)\varphi(z_n)) \\ G(z_n - J^{-1}(f,g)(z_n)\varphi(z_n)) \end{bmatrix}\right) \tag{21}$$

where,

$$J^{-1}(f,g)(z_n) = \begin{bmatrix} f_x(z_n) & f_y(z_n) \\ g_x(z_n) & g_y(z_n) \end{bmatrix}^{-1}.$$

Three different systems of equations are solved and both algorithms are able to locate the solutions. The second algorithm (21) proves to be far superior than the first (20) in finding the roots with fewer required iterations [**6**, p.550].

Newton's Method can also be used in optimization problems such as unconstrained minimization, equality constrained problems, convex programming, and interior methods. Kantorovich proposed a method similar to Newton's Method, the *Newton-Kantorovich* method, where $F(x) = 0$, having $F: X \rightarrow Y$, where X and Y are Banach spaces (normed vector spaces). This method can be read as

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \ k = 0, 1,2, \dots \qquad (22)$$

where $F'(x_k)$ is the Frechet derivative (derivative defined in Banach spaces) of the nonlinear operator $F(x)$ at point x_k [**5**, p. 1088]. These properties have a wide range of applications as lots of nonlinear problems exist such as; nonlinear integral equations, ordinary and partial differential equation, and variational problems.

A variation of Newton's Method is also presented to reduce the computational burden of finding the roots. Since the method requires calculating the derivative of a function, it may require huge processing power especially in complex vector functions.

A similar method

$$x_{k+1} = x_k - F'(x_0)^{-1}F(x_k), \qquad k = 0, 1, 2, \dots \qquad (23)$$

computes the derivative only at the first iteration and converges under similar conditions. The tradeoff of using this method is that it converges linearly, not quadratically [**5**, p 1088].

Modifications on Newton's Method have been created to increase the convergence rate over the years. These modified methods require an increase in function or derivative evaluations to perform the iterations. Therefore, the computational efficiency is decreased when increasing the convergence rate [**11**, p. 478].

An example of a modified method that has cubic convergence is

$$x_{n+1} = x_n - \frac{f\left(x_n + f(x_n)/(\lambda_n f(x_n) + f'(x_n))\right) - f(x_n)}{\lambda_n f(x_n) + f'(x_n)} \quad (24)$$

where,

$$\lambda_n \epsilon R, 0 \ \le |\ \lambda_n| \le 1, \quad n = 0, 1,2, \ldots$$

are parameters. The parameters are chosen such that $sign\left(\lambda_n f(x_n)\right) = sign(f'(x_n))$ where,

$$sign(x) = \begin{cases} 1 & if\ x \ge 0, \\ -1 & if\ x < 0. \end{cases}$$

This modification of Newton's Method eliminates the restriction of having $f'(x_n) \neq 0$ near the root of the function [**8,** p. 410-411].

For multiple roots, Newton's Method does not have quadratic convergence. As such, modifications for the method have been suggested for improving the convergence. The *Relaxed Newton's Method,*

$$x_{n+1} = x_n - \frac{mf(x_n)}{f'(x_n)}, \quad n = 0, 1, 2,3, \ldots \quad (25)$$

can be applied to a function which has multiple roots of order *m*. If the root is exactly of order *m*, the convergence is quadratic. If the root is of order *k*, the root can be either *attracting*, *repelling*, or *neutral*. If $m > 2k$, the root will be *repelling*; that is, the orbits of nearby points move away from the root. If $m < 2k$, the root will be *attracting*; that is, the orbits of nearby points will move towards this root, but the convergence to the root will be linear. If $m = 2k$, then the root is *neutral*; that is, nearby points could either slowly converge or diverge from this point [**9**, p. 257].

The *Newton's Method for Multiple Root,*

$$x_{n+1} = \ x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)}, \quad n = 0, 1,2,3, \ldots \quad (26)$$

is quadratically convergent for every root of f, but the tradeoff of using this method is that it involves the computation of the second derivatives therefore increasing the computational requirements [**9**, p. 258].

Other modifications such as the Collatz, Schröder, König, and Steffensen's methods are some of the many examples of variations of Newton's method. The flexibility of Newton's Method allows it to be modified in various ways in order to adapt to various problems [**9,** p. 258-260].

Chapter 6: Conclusion

Locating the roots of functions has played a major role in mathematics for some years now. To this day, students are still required to locate and describe where the roots are for certain functions. Although there are multiple methods to choose from, this paper discussed Newton's Method.

Newton's Method provides an easy and a simple solution for finding the roots of a function. If the roots are simple and the seed value is chosen within close proximity, the method has a quadratic convergence and is quite fast in finding the roots of a function. This method exploits the tangent lines to a function at certain points in order to find the roots. Each iteration of the method brings the value closer and closer to the real root. However, there are some cases for which Newton's Method does not converge to the root. The convergence of the function is determined by the function itself, or by the initial seed value used for the iteration. Such cases occur when the derivative of a function at a given point is zero, having a horizontal tangent line at that point. Other cases where the method fails are when the iterations of the values are thrown into an infinite loop or when the values of the seed of the iterations are diverging away from the root.

Newton's Method can be used for various real world applications. Since the method is simple and reliable in ways that it can be modified easily, it is the optimal choice for some applications which require root-finding algorithms. Golbabai and Javidi modify the method to find solutions to non-linear system of equations. The Newton-Kantorovich method is used for vector functions. The Relaxed Newton's Method and Newton's Method for Multiple Roots are used for functions having more than one root and are quite fast in doing it. Other modifications of Newton's Method also exist which will improve the convergence. The discovery of this method offers a great contribution to contemporary computations. Root-finding algorithms are required for complex computations, and Newton’s Method is a simple significant choice for accomplishing this.

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